On (sub)-Holonomicity of Some Modules and b-Functions (超函数と線型微分方程式 V)

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本記録においては、$\mathcal{D}$-Module の section u に対して、$\mathcal{N} = \mathcal{D}[s]f^s u$, $\mathcal{D}f^\alpha u$ $(\alpha \in \mathbb{C})$ を定義し、その性質をしらべる。この $\mathcal{N}$ は $\mathcal{D}[t,s]$-Module の構造をもち、t は injective になっている。$\mathcal{N}$ の b函数は常に存在し、その根によって、$\mathcal{D}u$ が holonomic の場合が重要であり、その場合 $\mathcal{D}f^\alpha u$ と $\mathcal{N}/(s-\alpha)\mathcal{N}$ の同型性も判別される。定理 1.17, 20, 21, 24 が重要である。又、reduced b-function についての事項もおさめた。詳細は下記の二論文、又それにつづくものに発表される。

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Chapter I Genelarities

In this chapter, we study the basic features of general $\mathcal{D}[t,s]$-Modules and b-functions associated with them, which are indispensable to later chapters. The author develop the general theory of such b-functions and Modules in [32].

§1. $\mathcal{D}[t,s]$ - Modules and b-functions.

Let $\mathbb{C}[t,s]$ be the associative algebra over $\mathbb{C}$ with generators s and t and defining relation

$$ts - st = t. \tag{1}$$

Set $\mathcal{D}[t,s] = \mathcal{D} \underset{\mathbb{C}}{\otimes} \mathbb{C}[t,s]$.

A $\mathcal{D}$- Module $\mathcal{M}$ is called a $\mathcal{D}[s]$ - Module (restectively $\mathcal{D}[t,s]$ - Module), if $\mathcal{M} \supset s\mathcal{M}$ (respectively $\mathcal{M} \supset s\mathcal{M}$, $\mathcal{M} \supset t\mathcal{M}$) holds. In this chapter, all Modules are $\mathcal{D}[t,s]$ - Modules unless otherwise stated. Since $t^{\nu}s = (s+\nu)t^{\nu}$ in view of (1), Ker t^{ν}, Coker t^{ν} and Im t^{ν} are $\mathcal{D}[t,s]$-Modules along with a given $\mathcal{D}[t,s]$-Module.

Definition 1.1 Let $\mathcal{L}$ be a $\mathcal{D}[s]$-Module. If $s \in \mathcal{E}nd_{\mathcal{D}}(\mathcal{L})$ has the non-zero minimal polynomial, we denote it by $d_{\mathcal{L}}(s)$, and say " $d_{\mathcal{L}}(s)$ exists." "b-functions" for a $\mathcal{D}[t,s]$-Module $\mathcal{N}$ are defined by $b_{\mathcal{N},\nu}(s) = d_{\mathcal{N}/t^{\nu}\mathcal{N}}(s)$, $\nu = 1,2,\cdots$.

Usually, $b_{\mathcal{N},1}$ is abbribiated as $b_{\mathcal{N}}$. As is easily seen, $b_{\mathcal{N},\nu}$ exist if and only if $b_{\mathcal{N}}$ exists.

It should be remarked that if $\mathcal{L}$ is a holonomic $\mathcal{D}[t,s]$-Module $d_{\mathcal{L}}(s)$ exists, since $\mathcal{E}nd_{\mathcal{D}}(\mathcal{L})_x$ $(x \in X)$ is finite dimensional and $\mathcal{E}nd_{\mathcal{D}}(\mathcal{L})$ is coherent [13].

Standard example of $\mathcal{D}[t,s]$ - Module is constructed as follows. Let f be a holomorphic function on $U \subset X$, let $\mathcal{L}$ be a coherent $\mathcal{D}$-Module and let u be its section over U. We denote the annihilator of u by $\mathcal{J}$, that is; $\mathcal{J} = \{Q \in \mathcal{D} \mid Qu=0\}$. Define the ideal $\mathcal{J}(s) \subset \mathcal{D}[s]$ by the condition that

$$P(s,x,D) \in \mathcal{J}(s) \quad \text{if and only if}$$
$$f^m P(s,x,D + \tfrac{s}{f}\,\mathrm{grad}\ f) \in \mathbb{C}[s] \otimes \mathcal{J}, \text{ for some } m.$$

We denote by $\mathcal{N}$ the Module $\mathcal{D}[s]/\mathcal{J}(s)$ and by $f^s u$ the class (1 mod $\mathcal{J}(s)$). $\mathcal{N} = \mathcal{D}[s]f^s u$ is a $\mathcal{D}[t,s]$-Module with actions of t and s given by,

$$t: P(s) \longmapsto P(s+1)f \quad , \qquad s: P(s) \longmapsto P(s)s .$$

The map t is injective in $\mathcal{N}$. In fact, if $P(s+1)f \in \mathcal{J}(s)$ then

$$f^m P(s+1,x,D+\frac{s}{f}\text{ grad } f)f = \sum Q_j s^j$$

for some m and $Q_j \in \mathcal{D}$. The left-hand side equals to

$$f^{m+1}P(s+1,x,D+\frac{s+1}{f}\text{ grad } f),$$

and the right-hand side can be rewritten in the form

$$\sum R_j (s+1)^j$$

for some $R_j \in \mathcal{D}$. Therefore,

$$f^{m+1}P(s,x,D+\frac{s}{f}\text{ grad } f) = \sum R_j s^j ,$$

which implies $P(s) \in \mathcal{J}(s)$.

The $\mathcal{L}$- Module $\mathcal{D} f^s u$ is coherent, and if u is a holonomic section, $\mathcal{L} f^s u$ is subholonomic (See [32]).

<u>Definition 1.2</u> With a non-zero polynomial $p(s)$, we associate a number $w(p) \in \mathbb{N}_0$ in the following manner ($w(p)$ is called the width of p.)

i) If $p(s) \in \mathbb{C}^*$ then $w(p) = 0$,

ii) If $p(s) = \prod_{i=0}^{k}(s+\alpha+i)^{\varepsilon_i}$, $\alpha \in \mathbb{C}$, $\varepsilon_0\varepsilon_k \neq 0$ then $w(p) = k+1$,

iii) If $p(s)$ has the form

$p(s) = \prod_{j=1}^{k} p_j(s)$, where each $p_j(s)$ is of the form in ii), $p_j(s) = \prod (s+\alpha_j+i)^{\varepsilon_i^{(j)}}$, and $\alpha_j \not\equiv \alpha_{j'}$ mod $\mathbb{Z}$ $(j \neq j')$; then $w(p) = \max_j w(p_j)$.

<u>Theorem 1.3</u> If $d_{\mathcal{L}}(s)$ exists, then $t^{w(d_{\mathcal{L}})}\mathcal{L} = 0$. Furthermore if we assume that t is injective or surjective, then $\mathcal{L} = 0$.

Proof) we have

$$0 = d_{\mathcal{L}}(s)\mathcal{L} \supset d_{\mathcal{L}}(s)t^{w(d_{\mathcal{L}})}\mathcal{L},$$

and by virtue of (1),

$$0 = t^{w(d_{\mathcal{L}})}d_{\mathcal{L}}(s)\mathcal{L} = d_{\mathcal{L}}(s+w(d_{\mathcal{L}}))t^{w(d_{\mathcal{L}})}\mathcal{L}.$$

It follows from the definition of $w(d_{\mathcal{L}})$ that

$$\text{g.c.d.}(d_{\mathcal{L}}(s), d_{\mathcal{L}}(s+w(d_{\mathcal{L}}))) = 1.$$

Hence the assertion follows. When t is injective or surjective, it is obvious that $\mathcal{L} = 0$. Q.E.D.

A coherent $\mathcal{E}$-Module $\mathcal{L}$ is called holonomic (resp. sub-holonomic) if $\mathcal{E}xt^i_{\mathcal{E}}(\mathcal{L}, \mathcal{E}) = 0$ for $i < n$ (resp. $i < n-1$).

This condition is equivalent to codim $\check{SS}(\mathcal{L}) \geq n$ (resp. codim $\check{SS}(\mathcal{L}) \geq n-1$). $\mathcal{L}$ is called purely subholonomic if $\mathcal{E}xt^{i}_{\mathcal{E}}(\mathcal{L},\mathcal{E}) = 0$ for $i \neq n-1$. It is known that for any coherent $\mathcal{E}$-Module, $\mathcal{E}xt^{n}_{\mathcal{E}}(\mathcal{L},\mathcal{E})$ (resp. $\mathcal{E}xt^{n-1}_{\mathcal{E}}(\mathcal{L},\mathcal{E})$) is holonomic (resp. sub-holonomic) and $\mathcal{E}xt^{i}_{\mathcal{E}}(\mathcal{L},\mathcal{E}) = 0$, $i > n$. Let W be an irreducible component of $\check{SS}(\mathcal{L})$. Then the multiplicity of $\mathcal{L}$ at a generic point x_0 of an irreducible component of $\check{SS}(\mathcal{L})$ can be defined (which is denoted by $m_{x_0}(\mathcal{L})$), and has the additivity, that is, if

$$0 \leftarrow \mathcal{L}_1 \leftarrow \mathcal{L}_2 \leftarrow \mathcal{L}_3 \leftarrow 0,$$

is an exact sequence of coherent $\mathcal{E}$-Modules, $m_{x_0}(\mathcal{L}_2) = m_{x_0}(\mathcal{L}_1) + m_{x_0}(\mathcal{L}_3)$.

Corollary 1.4 Let $\mathcal{M}$ be a sub-holonomic $\mathcal{E}[t,s]$-Module such that $t: \mathcal{M} \to \mathcal{M}$ is injective. Then, $\mathcal{M}$ is purely sub-holonomic.

Proof) Consider the exact sequence

$$0 \leftarrow \mathcal{M}/t\mathcal{M} \leftarrow \mathcal{M} \xleftarrow{t} \mathcal{M} \leftarrow 0.$$

Set $\mathcal{L} = \mathcal{E}xt^{n}_{\mathcal{E}}(\mathcal{M},\mathcal{E})$. Then $\mathcal{L}$ is holonomic and the long exact sequence of $\mathcal{E}xt$ gives us the surjection $\mathcal{L} \xrightarrow{t} \mathcal{L} \to 0$. Therefore $\mathcal{L} = 0$ by virtue of Theorem 1.3. Q.E.D.

Proposition 1.5 Upon the conditions in Corollary 1.4, $b_{\mathcal{M}}$ exists.

Proof) Consider an irreducible component W of $\check{SS}(\mathcal{M})$. Since t is injective, the multiplicity of $\mathcal{M}/t\mathcal{M}$ at a generic point of W vanishes. Therefore codim $\check{SS}(\mathcal{M}/t\mathcal{M}) \geq n$ which implies that $\mathcal{M}/t\mathcal{M}$ is holonomic. Thus $b_{\mathcal{M}}$ exists (and so does $b_{\mathcal{M},\nu}$, by the argument after Definition 1.1.). Q.E.D.

The conditions in Corrollary 1.4 are satified for $\mathcal{M} = \mathcal{D}[s]f^s u$, if one of the following two conditions holds.

i) f is arbitrary holomorphic function, u = 1.

ii) f is quasi-homogeneous, $\mathcal{D}u$ is holonomic.

In the present paper, we restrict ourselves to case i). We investigate case ii) in [32], where the detailed structure of $b_{\mathcal{M},\nu}(s)$ and the relation between $\mathcal{M}_\alpha$ and $\mathcal{D}f^\alpha u$ $(\alpha \in \mathbb{C})$ are also discussed. The existence of $b_{\mathcal{M}}(s)$ for $\mathcal{M} = \mathcal{D}[s]f^s u$ with general f and $\mathcal{D}u$ being holonomic can be derived from that of case ii), following the tecnique in of [14]. (See [32] §.).

以下は別の原稿よりとったので，以上と直接はつながらない。

A. General structure of $\mathcal{D}$[t,s]-Modules

In §1 we study the structure of $b_{\mathcal{N},\nu}(s)$ and define reduced b-functions. The relation between reduced b-function of $\mathcal{N}_1$ and that of a sub-Module $\mathcal{N}_2$ is studied in §2.

The key theorem is the following.

<u>Theorem 1.1</u> (Theorem 1.3 in [Y])

Let $\mathcal{L}$ be a $\mathcal{D}$[t,s]-Module such that $d_{\mathcal{L}}(s)$ exist. Then $t^{w(d_{\mathcal{L}})}\mathcal{L} = 0$.

Here, $w(d_{\mathcal{L}})$ is the width of $d_{\mathcal{L}}$. We recall its meaning and add some more definitions.

<u>Definition 1.2</u> For a non-zero polynomial $p(s)$, we associate a number $w(p)$, called the width of p, and polynomials $\hat{p}(s)$ and $\check{p}(s)$ in the following manner.

i) $p(s) \in \mathbb{C}^*$; $w(p) = 0$, $\hat{p}(s) = \check{p}(s) = 1$

ii) $p(s) = \prod_{i=0}^{k}(s+\alpha+i)^{\varepsilon_i}$, $\alpha \in \mathbb{C}$, $\varepsilon_0\varepsilon_k = 0$; $w(p) = k+1$, $\hat{p}(s) = (s+\alpha)^{\varepsilon_0}$, $\check{p}(s) = (s+\alpha+k)^{\varepsilon_k}$.

iii) $p(s) = \prod_{j=1}^{k} p_j(s)$, where each $p_j(s)$ is of the form in ii), $p_j(s) = \prod(s+\alpha_j+i)^{\varepsilon_i^{(j)}}$, and $\alpha_j \not\equiv \alpha_{j'}$ mod $\mathbb{Z}$ $(j \neq j')$; $w(p) = \max_j w(p_j)$, $\hat{p}(s) = \prod \hat{p}_j(s)$, $\check{p}(s) = \prod_j \check{p}_j(s)$.

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§1. Structure of $b_{\pi,\nu}(s)$.

We first note that $b_{\pi,\nu}(s)$ is of the significant structure. Given a rational function $p(s)$, we use the notation

$$[p(s)]_\nu = \prod_{i=0}^{\nu-1} p(s+i) \quad \nu > 0, \quad [p(s)]_0 = 1.$$

<u>Theorem 1.3</u> i) There are a rational function $\bar{b}_\pi(s)$, polynomials $\bar{b}'_\pi(s)$ and $c_\pi(s)$, unique up to a constant multiple, and $\nu_0 \in \mathbb{N}_0$, such that for $\nu \geq \nu_0$,

$$b_{\pi,\nu}(s) = [\bar{b}_\pi(s)]_\nu c_\pi(s+\nu) \tag{2}$$

$$= c_\pi(s)[\bar{b}'_\pi(s)]_\nu , \tag{3}$$

$$\bar{b}_\pi(s)c_\pi(s+1) = c_\pi(s)\bar{b}'_\pi(s). \tag{4}$$

ii) If $t : \pi \to \pi$ is injective, $\bar{b}_\pi(s)$ is also a polynomial, and for $\nu \leq \nu_0$ there are polynomials $c_{\pi,\nu}(s)$ and $c'_{\pi,\nu}(s)$ such that

$$b_{\pi,\nu}(s) = [\bar{b}_\pi(s)]_\nu c_{\pi,\nu}(s+\nu)$$

$$b_{\pi,\nu}(s) = [\bar{b}_\pi(s)]_\nu c_{\pi,\nu}(s+\nu)$$

$$= c'_{\pi,\nu}(s)[\bar{b}'_\pi(s)]_\nu,$$

$$c_{\pi,\nu}(s) \mid c_{\pi,\nu'}(s), \quad c'_{\pi,\nu}(s) \mid c'_{\pi,\nu'}(s) \quad \text{for } \nu \leq \nu'.$$

$$\hat{b}_\pi(s) \mid \bar{b}_\pi(s), \quad \check{b}_\pi(s) \mid \bar{b}'_\pi(s). \tag{5}$$

Moreover, it is possible to take $\nu_0 = w(b_\pi) - 1$, and the following relations hold.

$$c_\pi(s) \mid [\bar{b}_\pi(s)]_{\nu_0}, [\bar{b}'_\pi(s-\nu_0)]_{\nu_0}, \tag{6}$$

$$b_\pi(s) \mid [\bar{b}_\pi(s)]_{\nu_0+1}, [\bar{b}'_\pi(s-\nu_0)]_{\nu_0+1}, \tag{7}$$

$$w(c_\pi) \leq \nu_0. \tag{8}$$

Corollary 1.4 As easily seen, $\bar{b}_\pi(s)$ and $\bar{b}'_\pi(s)$ can be so determined that

$$\bar{b}_\pi(s) = b_{\pi,\nu+1}(s)/b_{\pi,\nu}(s+1),$$

$$\bar{b}'_\pi(s) = b_{\pi,\nu+1}(s-\nu)/b_{\pi,\nu}(s-\nu), \qquad \nu \geq \nu_0.$$

$\bar{b}_\pi(s)$ is called the reduced b-function of π. The special case of the part of this theorem is substantially due to M.Sato [2].

この証明は省略する.

Here after R, $\bar{R}$, $\bar{R}'$, R_ν and C denote the set of the roots of equations $b_{\mathfrak{N}}(s) = 0$, $\bar{b}_{\mathfrak{N}}(s) = 0$, $\bar{b}'_{\mathfrak{N}}(s) = 0$ $b_{\mathfrak{N},\nu}(s) = 0$ and $c_{\mathfrak{N}}(s) = 0$ respectively, when t is injective.

Proposition 1.5 $\quad R \supset \bar{R}$, $\bar{R}'$, C; $\quad R \cap (R+1) \supset C$

$$R_k + k = \bigcup_{i=1}^{k}(R+i) = \bigcup_{i=1}^{k}(\bar{R}+i) \cup C, \quad R_k = \bigcup_{i=0}^{k-1}(R-i) = \bigcup_{i=0}^{k-1}(\bar{R}'-i) \cup C.$$

Proof is straightforward.

We end this section by adding the following remarks when $t \in \mathcal{E}nd(\mathfrak{N})$ is not necessarily injective.

Definition 1.6 We define the $\mathcal{D}[t,s]$-Module $\tilde{\mathfrak{N}}$ by $\mathfrak{N} / \bigcup_{\nu \geq 1} \mathrm{Ker}\, t^\nu$. (Hence t is injective in $\tilde{\mathfrak{N}}$.)

We can prove $\bar{b}_{\mathfrak{N}}(s) = \frac{\tilde{c}(s)}{\tilde{c}(s+1)} \bar{b}_{\tilde{\mathfrak{N}}}(s)$, where $\tilde{c}(s) = c_{\mathfrak{N}}(s)/c_{\tilde{\mathfrak{N}}}(s)$ is a polynomial, and $\bar{b}'_{\mathfrak{N}}(s) = \bar{b}'_{\tilde{\mathfrak{N}}}(s)$. The proof is omitted.

Proposition 1.7 Let $0 \to \mathcal{L} \hookrightarrow \mathfrak{N} \to \mathfrak{N}' \to 0$ be an exact sequence of $\mathcal{D}[t,s]$-Modules, let $t \in \mathcal{E}nd_{\mathcal{D}}(\mathfrak{N}')$ be injective, and let $d_{\mathcal{L}}(s)$ exist. Then $\mathfrak{N}' \cong \tilde{\mathfrak{N}}$.

For, since $\mathfrak{N} \twoheadrightarrow \mathfrak{N}'$ and $t|_{\mathfrak{N}'}$ is injective, $\bigcup_\nu \mathrm{Ker}(t|_{\mathfrak{N}})^\nu \hookrightarrow \mathcal{L}$. On the other hand, $t^{w(d_{\mathcal{L}})}\mathcal{L} = 0$ by Theorem 1.1. Therefore, $\mathcal{L} = \mathrm{Ker}\, t^{w(d_{\mathcal{L}})} = \bigcup \mathrm{Ker}\, t^\nu$, and $\tilde{\mathfrak{N}} \cong \mathfrak{N}'$.

§2. b-functions for a sub-Module

In terms of $b_{\mathcal{M}}(s)$, we can estimate the b-function of a submodule of $\mathcal{M}$.

Theorem 1. 8 Let $\mathcal{M}_1$ be $\mathcal{D}[t,s]$-Module and let $\mathcal{M}_2$ be its submodule. Further assume 1° $t \in \mathcal{E}nd(\mathcal{M}_1)$ is injective, 2° $d_{\mathcal{M}_1/\mathcal{M}_2}(s)$ exists and 3° $b_{\mathcal{M}_1}(s)$ or $b_{\mathcal{M}_2}(s)$ exists. Then, $\deg \bar{b}_{\mathcal{M}_1} = \deg \bar{b}_{\mathcal{M}_2}$ (= d) and there are polynomials c(s) and c'(s), unique up to a constant multiple, such that

$$c(s)\ ,\ c'(s)\ \big|\ d_{\mathcal{M}_1/\mathcal{M}_2}(s), \tag{15}$$

$$c_{\mathcal{M}_1}(s)c'(s) = c_{\mathcal{M}_2}(s)c(s), \tag{16}$$

$$\bar{b}_{\mathcal{M}_1}(s) = \frac{c(s)}{c(s+1)}\bar{b}_{\mathcal{M}_2}(s),\ \bar{b}'_{\mathcal{M}_1}(s) = \frac{c'(s)}{c'(s+1)}\bar{b}'_{\mathcal{M}_2}$$

Corollary 1. 9

$$b_{\mathcal{M}_2}(s)\ \big|\ [b_{\mathcal{M}_1}(s)]_{\nu_0+1},\ b_{\mathcal{M}_1}(s)\ \big|\ [b_{\mathcal{M}_2}(s-\nu_0)]_{\nu_0+1}, \tag{18}$$

$$\bar{b}_{\mathcal{M}_2}(s)\ \big|\ [\bar{b}_{\mathcal{M}_1}(s)]_{\nu_0+1},\ \bar{b}_{\mathcal{M}_1}(s)\ \big|\ [\bar{b}_{\mathcal{M}_2}(s-\nu_0)]_{\nu_0+1}, \tag{19}$$

$$b_{\mathcal{M}_2}(s)\ \big|\ [\bar{b}_{\mathcal{M}_1}(s)]_{\nu_0+\nu'},\ b_{\mathcal{M}_1}(s)\ \big|\ [\bar{b}_{\mathcal{M}_2}(s-\nu_0)]_{\nu_0+\nu'}, \tag{20}$$

$$|\deg c_{\mathcal{M}_1} - \deg c_{\mathcal{M}_2}| \leqslant \nu_0 d, \tag{21}$$

where $\nu_0 = w(d_{\mathcal{M}_1/\mathcal{M}_2})$, $\nu' = \min(w(b_{\mathcal{M}_1}), w(b_{\mathcal{M}_2}))$.

Proof of Theorem 1.7) It follows from Thm.1.1 and condition 2 that $\pi_1 \supset t^{\nu_0}\pi_2$. Consider the following diagram for $\nu \geq \nu_0$,

$$\begin{tikzcd}[column sep=small]
 & & t^{\nu_0}\pi_1 \arrow[dr, phantom, "\supset" sloped] & & \\
\pi_1 \arrow[r, phantom, "\supset"] & \pi_2 \arrow[ur, phantom, "\supset" sloped] \arrow[dr, phantom, "\supset" sloped] & & t^{\nu}\pi_1 \arrow[r, phantom, "\supset"] & t^{\nu}\pi_2 \\
 & & t^{\nu-\nu_0}\pi_2 \arrow[ur, phantom, "\supset" sloped] & &
\end{tikzcd}$$

This immediately reads

$$\text{i)}\quad b_{2,\nu-\nu_0}(s) \mid b_{1,\nu}(s) \tag{22}$$

$$b_{1,\nu-\nu_0}(s+\nu_0) \mid b_{2,\nu}(s) \tag{23}$$

$$\text{ii)}\quad b_{2,\nu}(s) \mid b_{1,\nu}(s)d(s+\nu), \tag{24}$$

$$b_{1,\nu}(s) \mid d(s)b_{2,\nu}(s). \tag{25}$$

Here, we have used the notations, $b_{i,\nu}(s) = b_{\pi_i,\nu}(s)$, $c_i(s) = c_{\pi_i}(s)$, $d(s) = d_{\pi_1/\pi_2}(s)$. (22) and (23) tell us that the existence of b_1 and that of b_2 are equivalent. In particular, setting $\nu = \nu_0+1$, we have (18).

i) gives also,

$$(\nu - \nu_0)\deg b_2 + \deg c_2 \leq \deg b_1 + \deg c_1,$$

$$(\nu - \nu_0)\deg b_1 + \deg c_1 \leq \deg b_2 + \deg c_2,$$

and letting ν tend to infinity, we have $\deg b_1 = \deg b_2$. This implies (21).

Because of (22) and (23), we can assume, $\bar{b}_i(s) = \prod_{j=1}^{d}(s+n_j^{(i)})$, and $n_j^{(i)} \leq n_{j+1}^{(i)}$ for $n_j^{(i)} \in Z$. Setting $\nu \gg 0$ in formula (22), we have $s+n_1^{(2)} \mid [\bar{b}_1(s)]$, hence $n_1^{(2)} \geqslant n_1^{(1)}$. Similarly by (23), $n_1^{(1)} + \nu_0 \geq n_1^{(2)}$. Therefore $r_{1,\nu}(s) = [s+n_1^{(1)}]_\nu / [s+n_1^{(2)}]_{\nu-\nu_0}$ is a polynomial. Then the relation

$$[\prod_{j=2}^{d}(s+n_j^{(2)})]_{\nu-\nu_0}\, c_2(s+\nu-\nu_0) \mid r_{1,\nu}(s)[\prod_{j=2}^{d}(s+n_j^{(1)})]_\nu c_1(s+\nu),$$

for $\nu \gg 0$ yields $n_2^{(2)} \geqslant n_2^{(1)}$, and similarly, $n_2^{(1)} + \nu_0 \geqslant n_2^{(2)}$. Proceeding in this way, we have

$$n_j^{(1)} + \nu_0 \geqslant n_j^{(2)} \geqslant n_j^{(1)} \qquad j = 1,..,d. \qquad (26)$$

Set $c(s) = \prod_{j=1}^{d}[s+n_j^{(1)}]_{n_j^{(2)}-n_j^{(1)}}$. Then clearly $c(s)$ is a polynomial and the first of (17) holds. Uniqueness of $c(s)$ is obvious. We apply (17) to (25) and have, after cancellation,

$$c(s)\,c_1(s+\nu) \mid d(s)c_2(s+\nu)c(s+\nu),$$

taking $\nu \gg 0$, $c(s) \mid d(s)$.

State ments about $c'(s)$ can be proved analogously, and equation (17) applied to (2) and (3) gives (16). From equation (17), we have

$$c(s)[\bar{b}_2(s)]_\nu = [\bar{b}_1(s)]_\nu c(s+\nu). \qquad (27)$$

The definition of $c(s)$, together with (26) and (27) (with $\nu = \nu_0 + 1$) gives

(19). Analogously, (18), (26) and (27) (with $\nu = \nu_0 + \nu'$) prove (20).

Q.E.D.

Remark 1. $b_{\mathcal{M}_2}(s) \mid [b_{\mathcal{M}_1}(s)]_{\nu_0+1}$ (28)

holds even when t is not injective.

2. (27), (16) and (2) give

$$c(s)b_{2,\nu}(s) = b_{1,\nu}(s)c'(s+\nu). \tag{29}$$

3. Let h and k be the minimum and the maximum of the indices which satisfy $n_i^{(1)} < n_i^{(2)}$ respectively. Then by (15) $n_k^{(2)} - n_h^{(1)} \leq \nu_0$. It should be noted that this inequality improves (26).

<u>Theorem 1.10</u> Let X' and X be complex analytic and let $\pi : X' \to X$ be projective holomorphic map. For an $f(x) \in \mathcal{O}_X$, we set $f' = f \cdot \pi$. We assume $X' - \pi^{-1}(f^{-1}(0)) \cong X - f^{-1}(0)$. Then, $\mathcal{M} = \mathcal{D}_X[s]f^s$ is a sub-Module of $\mathcal{M}'' = \int \mathcal{M}'$, $\mathcal{M}' = \mathcal{D}_{X'}[s]f'^s$, and

$$b_{f,x}(s) \mid [b_{f',\pi^{-1}(x)}(s)]_{\nu_0+1},\ [\overline{b}_{f',\pi^{-1}(x)}(s)]_{\nu_0+w(b_{f'})}. \tag{30}$$

Here $\nu_0 = w(d\ \mathcal{M}''/\mathcal{M})$.

この証明は省略する.

B. Structure of $\mathcal{L}[s](f^s u)$

In the following sections, we investigate the structure of special $\mathcal{D}[t,s]$-Module $\mathcal{M} = \mathcal{L}[s](f^s u)$. It is to be proved that if u is a holonomic section, $\mathcal{D}f^s u$ is subholonomic and $\mathcal{D}f^\alpha u$ is holonomic. The characterization of reduced b-function is also given.

In the sequel, $\mathcal{M}$ always denote a $\mathcal{D}[t,s]$-Module $\mathcal{D}[s](f^s u)$ which is defined in §1 [Y]. Recall that the operation $t: P(s)(f^s u) \longmapsto P(s+1)f(f^s u)$ is <u>injective</u> in $\mathcal{M}$. We denote by $\mathcal{J}$ the annihilator of u. Basic concept and notations are same with S-K-K and [3]. Especially, a coherent $\mathcal{D}$-Module is called a System.

§3. Preliminary results on Systems

We define some general concepts and collect propositions which we shall need.

<u>Definition 1.11</u> For a system $\mathcal{L}$, we define

$$\mathrm{hol}(\mathcal{L}) = \begin{cases} \dim X - \operatorname{codim} \check{SS}(\mathcal{L}) & , \mathcal{L} \neq 0, \\ -\infty & , \mathcal{L} = 0. \end{cases}$$

Note that $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{L}, \mathcal{D}) = 0$ for $i < \dim X - \mathrm{hol}(\mathcal{L})$.

Definition 1.12 1. Let $\varphi: Y \to X$ be a holomorphic map and let $\mathcal{L}$ be a system on X. We define the induced Module of $\mathcal{L}$ on Y by

$$\varphi^* \mathcal{L} = \mathcal{E}_{Y \to X} \underset{\mathcal{E}_X}{\otimes} \mathcal{L} \quad .$$

2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be systems on X_1 and X_2, respectively. The product Module of $\mathcal{L}_1$ and $\mathcal{L}_2$ on $X_1 \times X_2$ is defined by:

$$\mathcal{L}_1 \hat{\otimes} \mathcal{L}_2 = \mathcal{E}_{X_1 \times X_2} \underset{\mathcal{E}_{X_1} \otimes \mathcal{E}_{X_2}}{\otimes} (\mathcal{L}_1 \underset{\mathbb{C}}{\otimes} \mathcal{L}_2).$$

3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be systems on X. The product Module of them on X is defined to be

$$\mathcal{L}_1 \boxtimes \mathcal{L}_2 = \Delta^*(\mathcal{L}_1 \hat{\otimes} \mathcal{L}_2),$$

where $\Delta : X \to X \times X$ is a diagonal embedding.

For the Definition 1.12, 1. and 2. and the following Theorem, we refer the reader to S-K-K and M.Kashiwara [3],[ ].

Theorem 1.13 1. Assume that for $V \subset P^*Y$, the map induced from the canonical projection is proper.

$$\rho^{-1}(V) \cap \varpi^{-1}(SS(\mathcal{L})) \longrightarrow V.$$

Then, $\varphi^* \mathcal{L}$ is a system on Y and the following isomorphism holds

$$\varphi^* \mathbb{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{D}_X)[\dim X] \xleftarrow{\sim} \mathbb{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\varphi^*\mathcal{L}, \mathcal{D}_Y)[\dim Y].$$

2. $\mathcal{L}_1 \hat{\otimes} \mathcal{L}_2$ is always a system and $(\mathcal{L}_1, \mathcal{L}_2) \mapsto \mathcal{L}_1 \hat{\otimes} \mathcal{L}_2$ is an exact functor.

3. If $\check{SS}(\mathcal{L}_1) \cap \check{SS}(\mathcal{L}_2) \subset X$, $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ is a system.

Statement 3 is derived from 1 and 2 easily.

Proposition 1.14 Upon the conditions in Theorem 1.13 ,

1. $\mathrm{hol}\,(\varphi^*\mathcal{L}) \leqq \mathrm{hol}\,(\mathcal{L})$

2. $\mathrm{hol}\,(\mathcal{L}_1 \hat{\otimes} \mathcal{L}_2) = \mathrm{hol}\,(\mathcal{L}_1) + \mathrm{hol}\,(\mathcal{L}_2)$,

3. $\mathrm{hol}\,(\mathcal{L}_1 \boxtimes \mathcal{L}_2) \leqq \mathrm{hol}\,(\mathcal{L}_1) + \mathrm{hol}\,(\mathcal{L}_2)$.

Since this is an easy Corrollary of Theorem 1.13, we omit the proof.

We note that Prof. Bernstein considered above theorems under a little different situation in of [+]. The notation $\boxtimes$ is barrowed from it.

4. Holonomicity and subholonomicity of some Modules

In this section, we study the structure of $\mathcal{D}[s]f^s u$ and $\mathcal{D}f^{\alpha}u$ when $\mathcal{D}u$ is holonomic.

We define the M:dules $\mathcal{N}_\alpha$ and $\mathcal{D}f^\alpha u$ for $\alpha \in \mathbb{C}$ as follows.

Definition 1.15.

$$\mathcal{N}_\alpha = \mathcal{N}/(s-\alpha)\mathcal{N} .$$

We sue the notation

$$\mathcal{J}(\alpha) = \{ P \in \mathcal{D} \mid P=Q(\alpha) \text{ for some } Q(s) \in \mathcal{J}(s) \} .$$

Then $\mathcal{N}_\alpha$ is isomorphic to $\mathcal{D}/\mathcal{J}(\alpha)$. Let $v \in \mathcal{N}$. Then, v mod $(s-\alpha)\mathcal{N}$ is denoted by $(v)_\alpha$. Especially, $(f^s u)_\alpha$ is the class 1 mod $\mathcal{J}(\alpha)$.

We define

$$\mathcal{J}_\alpha = \{ P \in \mathcal{D} \mid f^m P(x, D+\frac{\alpha}{f} df) \in \mathcal{J} \text{ for some } m. \}$$

Consider the Module $\mathcal{D}/\mathcal{J}_\alpha$ and denote 1 mod $\mathcal{J}_\alpha$ by $f^\alpha u$. Thus $\mathcal{D} f^\alpha u = \mathcal{D}/\mathcal{J}_\alpha$.

We also define

$$\mathcal{J}^{(0)} = \mathcal{J}(s) \cap \mathcal{D} .$$

The following inclusions hold.

$$\mathcal{J}^{(0)} \subset \mathcal{J}(\alpha) \subset \mathcal{J}_\alpha .$$

Proposition 1.16 Ideals $\mathcal{J}^{(0)}$, and $\mathcal{J}_\alpha$ are coherent.

Proof. The proof relies on the following theoremof M.Kashiwara.

"Let $\mathcal{I}$ be an ideal of $\mathcal{D}$ with filtration:
$\mathcal{I} = \bigcup_m \mathcal{I}_m$, $\mathcal{D}^{(h)}\mathcal{I}_m \subset \mathcal{I}_{m+h}$. In order to be coherent for $\mathcal{I}$ over $\mathcal{D}$, it is necessary and sufficient that each $\mathcal{I}_m$ is coherent over $\mathcal{O}$."

From this, the coherency of $\mathcal{J}^{(0)}$ and $\mathcal{J}_\alpha$ follows.

Q.E.D.

Thus we have three systems with canonical surjections.

$$\mathcal{D}f^s u \to \mathcal{N}_\alpha \to 0, \qquad \mathcal{N}_\alpha \to \mathcal{D}f^\alpha u \to 0.$$

We study (sub-)holonomicity of these Modules in the following.

Theorem 1.17 ... is subholonomic, when $\mathcal{D}u$ is holonomic.

Proof. Since $\mathcal{D}u$ is holonomic, M.Kashiwara's theorem in [5] says that $\check{SS}(\mathcal{D}u) \subset \bigcup T^*_{X_j} X$ for some Whitney stratification $X = \bigcup X_j$. We first prove

Lemma 1.13 $\mathcal{D}f^s \boxtimes \mathcal{D}u$ is subholonomic outside $f^{-1}(0)$.

Proof. It is sufficient to show $\check{SS}(\mathcal{D}f^s) \cap \check{SS}(\mathcal{D}u) \subset X$ outside $f^{-1}(0)$ by 3. of Theorem 1.14. We refine the stratification, if necessary, such that each X_j is contained in $f^{-1}(0)$ or disjoint to it. Assume that there exists (x_0, ξ_0) which has the following properties:* $x_0 \in f^{-1}(0)$, and there is an analytic path $x(t)$ in some X_j such that $x(0) = x_0$, $(x(t), \xi(t)) \in W$ for $t > 0$ and $\lim_{t \to 0} \xi(t) = \xi_0$. Since the tangent of the curve $x = x(t)$ is $(\dot{x}_1(t), \ldots, \dot{x}_n(t))$, we have

$$0 = \sum_{i=1}^{n} \dot{x}_i(t) \frac{f_i}{f} = \frac{d}{dt} f(x(t))$$

from the definition of W. Therefore, the path $x = x(t)$ is included in $f^{-1}(0)$, and so is X_j. q.e.d.

Owing to the canonical surjection and injection

$$\mathcal{D}f^s \boxtimes \mathcal{D}u \hookleftarrow \mathcal{D}(f^s \boxtimes u) \to \mathcal{D}f^s u \to 0,$$

$\mathcal{D}f^s u$ is subholonomic outside $f^{-1}(0)$. We use the argument in of [3]. Take the subholonomic part of $\mathcal{D}f^s u$ and denote it by $\mathcal{L}$. Lemma 1.18 shows that the support of the Module $\mathcal{D}f^s u/\mathcal{L} = \mathcal{D}\overline{f^s u}$ is contained in $f^{-1}(0)$. Therefore, considering the coherent $\mathcal{O}$-Module $\mathcal{O}\overline{f^s u}$, we have ~~an~~ a natural number k such that $f^k \cdot f^s u \in \mathcal{L}$. Since $\mathcal{D} f^k \cdot f^s u$ and $\mathcal{D} f^s u$ are isomorphic, the subholonomicity of $\mathcal{D} f^s u$ is derived from that of $\mathcal{D} f^k \cdot f^s u$. Q.E.D.

We note thatthe holonomicity of $\mathcal{D}f^s u/\mathcal{D}f^k \cdot f^s u$ is an easy consequence of the above theorem and injectivity of t, considering multiplicity of each Module in the following exact sequence along irreducible components of $\check{SS}(\mathcal{D}f^s u)$.

$$0 \to \mathcal{D}f^s u \xrightarrow{t^k} \mathcal{D}f^s u \to \mathcal{D}f^s u/\mathcal{D}f^k \cdot f^s u \to 0.$$

When f is quasi-homogeneous, $\mathcal{N} \cong \mathcal{D}/\mathcal{J}^{(0)}$ and hence subholonomic. Thus $b_{\mathcal{N}}(s)$ exists, by Proposition 1.5 in [Y] . In the general cases, we use the tecnique of adding a parameter. Define $f'(t,x) = tf(x)$. Then $\mathcal{N}' \cong \mathcal{D}_{\mathbb{C}\times X}/\mathcal{J}^{(0)}_{f'}$ and hence there exists $b'(s)$ and $Q(t,x,D_t,D_x)$ such that

$$Q(t,x,D_t,D_x)f'^{s+1}u = b'(s)f'^{s}u.$$

Let $Q_0(tD_t,x,D_x)D_t = \sum_j a_j(x,D_x)(tD_t)^j \cdot D_t$ be the homogeneous part of degree -1 in t of Q. Then, defining P by

$$P(s,x,D) = Q_0(s,x,D),$$

we have

$$P(s,x,D)f^{s+1}u = \frac{b'(s)}{s+1}f^{s}u$$

Thus b-function always exists. We denote by R the set of roots of the equation $b(s)=0$.

Theorem 1.20 $\mathcal{D}f^{\alpha}u$ is holonomic, when $\mathcal{D}u$ is holonomic.

Proof) As in the proof of Theorem 1.17, one can see that $\mathcal{D}f^{k}(f^{\alpha}u)$ is holonomic for sufficiently large k. Then the following diagram proves the holonomicity of $\mathcal{D}f^{\alpha}u$.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{D}f^{k}\cdot f^{s}u \arrow[r, hook] \arrow[d] & \mathcal{D}f^{s}u \arrow[r] \arrow[d] & \mathcal{D}f^{s}u/\mathcal{D}f^{k}\cdot f^{s}u \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{D}f^{k}\cdot f^{\alpha}u \arrow[r, hook] \arrow[d] & \mathcal{D}f^{\alpha}u \arrow[r] \arrow[d] & \mathcal{D}f^{\alpha}u/\mathcal{D}f^{k}\cdot f^{\alpha}u \arrow[r] \arrow[d] & 0. \\
& 0 & 0 & 0 &
\end{tikzcd}$$

<u>Theorem 1.21</u> $\mathcal{M}_\alpha \xrightarrow{\sim} \mathcal{D} f^\alpha u$ if and only if $\alpha \notin R + \mathbb{N}$.

Proof. Let $P \in \mathcal{J}_\alpha$, ord $P = m$. Then, there is $Q(s) \in \mathcal{D}[s]$ such that

$$Pf^m \equiv (s+m-\alpha)Q(s,x,D) \quad \text{mod } \mathcal{J}(s). \tag{40}$$

To prove (40) , we prepare

<u>Lemma 1.22</u> For any $R \in \mathcal{E}$, ord $R = m$, the following equality holds for some $S(s,x,D) \in \mathcal{E}[s]$ with $\text{ord}^T S < m$.

$$\left\{R(x, D+\tfrac{s}{f}\, df) - R(x, D+\tfrac{\gamma}{f}\, df)\right\} f^m = (s-\gamma)S(s,x,D+\tfrac{s}{f}\, df).$$

Proof. The proof is carried out by induction on m. When $m=0$, the result is trivial. Let $m \geqslant 1$. For the simplicity of the notation, we explain the case of one variable. General case is similar.

By the hypothesis of induction,

$$\left\{(D+s\tfrac{f'}{f})^{m-1} - (D+\gamma\tfrac{f'}{f})^{m-1}\right\} f^{m-1} = (s-\gamma)Q_{m-2}(s,x,D+s\tfrac{f'}{f}).$$

Then,

$$\left\{(D+s\tfrac{f'}{f})^m - (D+\gamma\tfrac{f'}{f})^m\right\} f^m$$

$$= \left\{(D+s\tfrac{f'}{f})^{m-1} - (D+\gamma\tfrac{f'}{f})^{m-1}\right\} (D+s\tfrac{f'}{f})f^m + (s-\gamma)(D+\gamma\tfrac{f'}{f})^{m-1}f'f^{m-1}$$

$$= (s-\gamma)Q_{m-2}(s,x,D+s\tfrac{f'}{f})(f(D+s\tfrac{f'}{f})+mf') + (s-\gamma)Q'_{m-1}(s,x,D+s\tfrac{f'}{f})f'$$

where

$$Q'_{m-1}(s,x,D) = (fD+(\gamma-s+1)f')(fD+(\gamma-s+2)f')..(fD+(\gamma-s+m-1)f').$$

This yields the case m. q.e.d.

We apply this lemma for $R = P$, $\gamma = \lambda - m$. Then, we have

$$P(x,D+\tfrac{s}{f}\,df)f^m - f^m P(x,D+\tfrac{\alpha}{f}\,df) = (s+m-\alpha)Q(s,x,D+\tfrac{s}{f}\,df),$$

which proves (40).

<u>Lemma 1.23</u> $t^m \mathcal{M} \cap (s+m-\lambda)\mathcal{M} \subset (s+m-\lambda)t^m\mathcal{M}$.

Because of the condition on λ, $(s+m-\lambda)$ is not a factor of $b_{\mathcal{M},m}(s)^*$. Hence we have an isomorphism

$$\mathcal{M}/t^m\mathcal{M} \xrightarrow[\sim]{s+m-\alpha} \mathcal{M}/t^m\mathcal{M}. \tag{41}$$

Now take an element $v = (s+m-\alpha)w \in t^m\mathcal{M} \cap (s+m-\lambda)\mathcal{M}$. If we consider $w \bmod t^m\mathcal{M}$ in the left-hand side of (41), it turns out to be 0 in the right-hand side. Hence $w \in t^m\mathcal{M}$, that is $v \in (s+m-\lambda)t^m\mathcal{M}$. q.e.d.

Owing to this lemma and (40), we obtain

$$Pf^m \equiv (s+m-\lambda)Q'(s,x,D)f^m \qquad \bmod \mathcal{J}(s),$$

Note that $R + \mathbb{N} = (\bar{R} + \mathbb{N}) \cup C$.

and hence

$$P \equiv (s-\alpha)Q'(s-m,x,D) \qquad \text{mod } \mathcal{J}(s),$$

which proves the "if part".

("only if" part) Suppose $\alpha \in R + \mathbb{N}$. Then, $^{\exists}\nu > 0$, such that $b_{\mathcal{N},\nu}(\alpha-\nu) = 0$. It follows from the definition of $b_{\mathcal{N},\nu}(s)$ that there exists $P_\nu(s) \in \mathcal{D}[s]$, such that

$$P_\nu(s+\nu)f^\nu \equiv b_{\mathcal{N},\nu}(s) \qquad \text{mod } \mathcal{J}(s).$$

Therfore $P_\nu(\alpha) \in \mathcal{J}_\alpha$. If $\mathcal{N}_\alpha \simeq \mathcal{D}f^\alpha$ were valid, we should have

$$P_\nu(\alpha) = Q(s) + (s-\alpha)R(s), \ ^{\exists}Q(s) \in \mathcal{J}(s).$$

Then, if we set $R_\nu(s) = (P_\nu(s) - P_\nu(\alpha))/(s-\alpha) + R(s)$,

$$R_\nu(s+\nu)f^\nu \equiv b_{\mathcal{N},\nu}(s)/(s+\nu-\alpha) \quad . \quad \text{mod } \mathcal{J}(s)$$

This contradicts the minimality of $b_{\mathcal{N},\nu}(s)$. Q.E.D.

There is a canonoical map

$$\tau : \mathcal{N}_{\alpha+1} \to \mathcal{N}_\alpha, \qquad (f^s u)_{\alpha+1} \mapsto f\cdot(f^s u)_\alpha .$$

As a map between $\mathcal{D}[s]$-Modules, this is $Q(s)\bar{1} \mapsto Q(s+1)f\cdot\bar{1}$. Since $(\mathcal{J}(s+1) + \mathcal{D}[s](s-\alpha))f \subset \mathcal{J}(s) + \mathcal{D}[s](s-\alpha)$, this map is well-defined. There is also a map $\mathcal{D}f^{\alpha+1}u \to \mathcal{D}fu$, defined by $f^{\alpha+1}u \longrightarrow f\cdot f^{\alpha}u$. These maps are compatible with the surjection (35).

<u>Theorem 1.24</u> i) $\mathcal{N}_{\alpha+1} \xrightarrow{\sim} \mathcal{N}_{\alpha}$ if and only if $\alpha \notin R$.

ii) When $\alpha \notin R$, $\mathcal{D} f^{\alpha+1}u \xrightarrow{\sim} \mathcal{D} f^{\alpha}u$.

Proof) We first prove "if part" of i) and ii). If $\alpha \notin R_f$, we can define the map $\rho : \mathcal{N}_{\alpha} \to \mathcal{N}_{\alpha+1}$, by $\bar{1} \mapsto b(\alpha)^{-1}P(\alpha)\bar{1}$. As a map between $\mathcal{D}[s]$-Modules, ρ is $R(s)\bar{1} \mapsto b(\alpha)^{-1}R(s-1)P(\alpha)\bar{1}$. Then this is a well-defined homomorphism, since if $R(s) \in \mathcal{J}(s) + \mathcal{D}[s](s-\alpha)$, $R(s-1)P(\alpha)f^{s}u \equiv 0 \mod (s-\alpha-1)\mathcal{D}[s]\cdot f^{s}u$. Similarly,

$$Q(s)fP(\alpha)f^{s}u = Q(s)f\{P(s-1) + (P(\alpha)-P(s-1))\}f^{s}u$$
$$\equiv Q(s)b(s-1)f^{s}u$$
$$\equiv b(\alpha)Q(s)f^{s}u \quad \mod (s-\alpha-1)\mathcal{D}[s]f^{s}.$$

Therefore, $\rho\tau(Q(s)\bar{1}) = \rho(Q(s+1)f\cdot\bar{1}) = b(\alpha)^{-1}Q(s)fP(\alpha)\bar{1} = Q(s)\bar{1}$. Analogously, $\tau\rho(R(s)\bar{1}) = R(s)\bar{1}$. Thus, ρ is the inverse of τ. The proof of ii) can be given in the same manner.

("only if" part) $\mathcal{N}_{\alpha+1} \simeq \mathcal{N}_{\alpha}$ implies

$$(\mathcal{J}(s+1) + \mathcal{D}[s](s-\alpha))f = \mathcal{J}(s) + \mathcal{D}[s](s-\alpha).$$

Hence, if $R(s)f^{s}u = (s-\alpha)Q(s)f^{s}u$, then there is $Q'(s)$ such that $R(s)f^{s}u = (s-\alpha)Q'(s)f\cdot f^{s}u$. Therefore, if $\alpha \in R$ were valid, the relation

$$P(s)f\cdot f^s u = (s-\alpha)\frac{b(s)}{s-\alpha}f^s u = (s-\alpha)Q'(s)f\cdot f^s u$$

shows $Q'(s)f\cdot f^s u = \frac{b(s)}{s-\alpha}f^s u$. This contradicts the minimality of b(s). Q.E.D.

Corollary 1.25 i) When $\alpha \notin R + \mathbb{N}$, the following commutative diagram exists for any $k \in \mathbb{N}_0$.
When $\alpha \notin R + \mathbb{Z}$, it holds for $k \in \mathbb{Z}$.

$$\begin{array}{ccc} \mathcal{N}_\alpha & \xrightarrow{\sim} & \mathcal{N}_{\alpha-k} \\ \downarrow\wr & & \downarrow\wr \\ \mathcal{D}f^\alpha u & \xrightarrow{\sim} & \mathcal{D}f^{\alpha-k}u \end{array}$$

ii) $\varinjlim \mathcal{N}_{\alpha-k} \cong \varinjlim \mathcal{D} f^{\alpha-k}u$ is holonomic for $\forall \alpha \in \mathbb{C}$.

Proof) i) is a direct consequence of Theorems 1.24 and 1.21.
ii) follows from i).

§5. Reduced b-functions

We can realize a reduced b-function as a b-function of some $\mathcal{D}[t,s]$-Modules. The characterization of these Modules are also given. We are indebted to M.Sato[2] for basic ideas in this section.

Definition 1.26

$$\mathcal{M}_{\#} = \bigcup_{\nu\geq 0} [\bar{b}(s-\nu)]_{\nu} t^{-\nu}\mathcal{M},$$

$$\mathcal{M}^{\#} = \{v(s) \in \bigcup_{\nu\geq 0} \mathcal{D}[s]t^{-\nu}\mathcal{M} \mid \exists m, [\bar{b}(s)]_m v(s) \in t^m \mathcal{M}_{\#}\}.$$

Proposition 1.27

i) $\mathcal{M}_{\#}$ and $\mathcal{M}^{\#}$ are $\mathcal{D}[t,s]$-Modules. If $\mathcal{M}$ is coherent, $\mathcal{M}_{\#}$ is also coherent.

ii) $b_{\mathcal{M}_{\#}} = b_{\mathcal{M}^{\#}} = \bar{b}$.

Proof) i) $\mathcal{M}_{\#}$ and $\mathcal{M}^{\#}$ are easily seen to be $\mathcal{D}[t,s]$-Modules. To see the coherency of $\mathcal{M}_{\#}$, we use the operators $P_{\nu}(s)$ which satisfy

$$P_{\nu}(s+\nu)f^{\nu} \equiv b_{\mathcal{M},\nu}(s) \quad \text{mod } \mathcal{J}(s).$$

Since $\mathcal{D}[s]$ is a noetherian ring, there is $m \in \mathbb{N}$ such that

$$P_m(s) + A_1(s)P_{m-1}(s) + \cdots + A_m(s)P_0(s) = 0.$$

for some $A_\nu(s) \in \mathcal{L}[s]$. Since

$$P_h(s)P_n(s+n) \equiv c(s)P_{h+n}(s+n) \quad \bmod \mathcal{J}(s+n),$$

multiplying $P_n(s+n)$ from the right, cancelling $c(s)$ and rewriting s to s-n, we have

$$P_{m+n}(s) + A_1(s-n)P_{m+n-1}(s) + \dots + A_m(s-n)P_n(s) \equiv 0 \bmod \mathcal{J}(s).$$

Therefore, $\mathfrak{N}_{\#} = \bigcup_{\nu=0}^{m-1} [\bar{b}(s-\nu)]_\nu t^{-\nu}\mathfrak{N} \subset t^{-m+1}\mathfrak{N}$.

ii) Obviously, $[\bar{b}(s)]_\nu \mathfrak{N}_{\#} \subset t^\nu \mathfrak{N}_{\#}$. Set $\bar{b}_\nu(s) = b_{\mathfrak{N}_{\#},\nu}(s)$. If $\bar{b}_\nu(s) \neq [\bar{b}(s)]_\nu$ for some ν, there is $k' < k = \deg \bar{b}(s)$, such that $\deg \bar{b}_\nu(s) < \nu k'$ for $\nu \gg 0$. But the following diagram shows $\nu k \leqslant (\nu+m-1)k'$. This is a contradiction. Thus we proved $b_{\mathfrak{N}_{\#}} = \bar{b}$.

$$\begin{array}{ccccc} 0 & & & & \\ \downarrow & & & & \\ \mathfrak{N}/t^{\nu+m-1}\mathfrak{N}_{\#} & \longrightarrow & \mathfrak{N}/t^\nu\mathfrak{N} & \longrightarrow & 0 \\ \downarrow & & & & \\ \mathfrak{N}_{\#}/t^{\nu+m-1}\mathfrak{N}_{\#} & & & & \end{array}$$

It follows from the definition of $\mathfrak{N}^{\#}$ that $\mathfrak{N}^{\#} \supset \mathfrak{N}_{\#}$ and $\bar{b}(s)\mathfrak{N}^{\#} \subset t\mathfrak{N}^{\#}$. Set $\bar{\bar{b}}(s) = b_{\mathfrak{N}^{\#},1}(s)$ and assume $\neq \bar{b}(s)$. Then or $v(s) \in \mathfrak{N}_{\#}$, $\bar{\bar{b}}(s)v(s) \in t\mathfrak{N}^{\#}$ yields $[\bar{b}(s)]_m \bar{\bar{b}}(s-1)v(s-1) \in t^m\mathfrak{N}_{\#}$.
This relation is equivalent to $\bar{\bar{b}}(s)[b(s+1)]_m v(s) \in t^{m+1}\mathfrak{N}_{\#}$.
Since $\mathfrak{N}_{\#}$ is finitely generated over $\mathcal{L}[s]$, we see that $b_{\mathfrak{N}_{\#},m}(s)$ is a strict divisor of $[\bar{b}(s)]_m$ for sufficiently large m. That is a contradiction. Hence $b_{\mathfrak{N}^{\#}}(s) = \bar{b}(s)$. Q.E.D.

It is not for certain whether $\mathcal{N}^{\#}$ is coherent or not when $\mathcal{N}$ is coherent. We have, however, the following characterization.

<u>Theorem 1.28</u> Let $\mathcal{N}'$ be a $\mathcal{D}[t,s]$-Module satisfying,

$t^{-k}\mathcal{N} \supset \mathcal{N}' \supset \mathcal{N}$ for some k. Then $b_{\mathcal{N}'}(s) = \bar{b}(s)$ if and only if

$$\mathcal{N}^{\#} \supset \mathcal{N}' \supset \mathcal{N}_{\#} .$$

Proof) ("only if" part) Since $b_{\mathcal{N}'}(s) = \bar{b}(s)$, we have relations

$$\mathcal{N}' \supset \bar{b}(s-1)t^{-1}\mathcal{N}' \supset \bar{b}(s-1)\bar{b}(s-2)t^{-2}\mathcal{N}' \supset [\bar{b}(s-h)]_h t^{-h}\mathcal{N}' .$$

Therefore,

$$\mathcal{N}' \supset \bigcup_{h\geq 0} [\bar{b}(s-h)]_h t^{-h}\mathcal{N}' \supset \bigcup_{h\geq 0} [\bar{b}(s-h)]_h t^{-h}\mathcal{N} = \mathcal{N}_{\#} .$$

Then the following diagram

$$\mathcal{N}' \supset \mathcal{N}_{\#} \supset t^m\mathcal{N}' \supset t^m\mathcal{N}_{\#} \tag{46}$$

shows that $d_{\mathcal{N}'/t^m\mathcal{N}_{\#}}(s)$ divides both $d'(s)[\bar{b}(s)]_m$ and $[\bar{b}(s)]_m d'(s+m)$ (where we set $d'(s) = d_{\mathcal{N}'/\mathcal{N}_{\#}}$), and hence one of $[\bar{b}(s)]_m$ for $m \gg 0$. But $[\bar{b}(s)]_m$ is the best possible for the pair $\mathcal{N}_{\#} \supset t^m\mathcal{N}_{\#}$. Therefore, $d_{\mathcal{N}'/t^m\mathcal{N}_{\#}}(s) = [\bar{b}(s)]_m$. Thus the definition of $\mathcal{N}$ proves $\mathcal{N}^{\#} \supset \mathcal{N}'$.

("if" part) Consider the following diagram for $m \gg 0$.

$$\mathcal{N}^{\#} \supset \mathcal{N}' \supset \mathcal{N}_{\#} \supset t^m \mathcal{N}' \supset t^m \mathcal{N}_{\#} .$$

Then the definition of $\mathcal{N}^{\#}$ implies $[\bar{b}_{\mathcal{N}'}(s)]_m c_{\mathcal{N}'}(s+m) \mid [\bar{b}(s)]_m$. On the other hand, equation (17) of Theorem 1.8 shows $\bar{b}_{\mathcal{N}'}(s) = (c'(s)/c'(s+1))\bar{b}(s)$. From these formulae, we have $c' = c_{\mathcal{N}'} = 1$, and then $b_{\mathcal{N}'}(s) = \bar{b}(s)$. Q.E.D.

Corollary 1.29 Assume that $w(\bar{b}) = 1$ in addition to the condition on $\mathcal{N}'$ in Theorem 1.28. Then, $b_{\mathcal{N}'} = \bar{b}$, if and only if $\mathcal{N}' = \mathcal{N}_{\#}$.

Proof) The "if" part is trivial. Consider the diagram (46) when $b_{\mathcal{N}'} = \bar{b}$. $d_{\mathcal{N}'/t^m\mathcal{N}_{\#}}(s) = [\bar{b}(s)]_m$ is shown in the proof of Theorem 1.28. Therefore, $d'(s) = d_{\mathcal{N}'/\mathcal{N}_{\#}}(s)$ and $d'(s+m)$ are divisors of $[\bar{b}(s)]_m$ for large m. Since $w(\bar{b}) = 1$, this is actually possible only when $d'(s) = 1$, that is, $\mathcal{N}' = \mathcal{N}_{\#}$. Q.E.D.